ERTH 455 / GEOP 555
Geodetic Methods for Understanding Earth's Surface Deformation
Lecture 05: Position Estimation with Pseudoranges

1 Introduction

Recall that a **range** describes the geometric distance between two points; here a satellite and a receiver. This could be inferred by measuring the transit time, τ, of a signal that travels from satellite to receiver at the speed of light, c. However, what the GPS receiver can provide is a range that biased by clock error effects, path delays due to ionosphere and atmosphere impacts and other error sources, which is why it's called a **pseudorange**. For this derivation we will carry the error terms along and assume that they can be effectively dealt with separately from the core of the positioning problem.

2 Pseudorange Measurement Model

The pseudorange from receiver u to satellite s, $\rho^{(s)}$, can be expressed as:

$$\rho^{(s)} = r^{(s)} + c(\delta t_u - \delta t^{(s)}) + I + T + \epsilon \quad (1)$$

where $r^{(s)}$ is the true range to satellite s, c remains the speed of light, δt_u is the receiver clock bias, $\delta t^{(s)}$ is clock bias of satellite s and I, T are ionospheric and tropospheric delays. The last term, ϵ, captures unmodeled effects, such as multipath, measurement errors, etc. Note that subscripts (e.g., u) reflect receiver specific values, while superscripts identify individual satellites; these are not powers of (s)!

3 Geometric Range

We have two ways to express the geometric range. One was mentioned above as the transit time, τ multiplied by the speed of light, c. However, this does not include the receiver position, which we would like to estimate. Instead, we can say:

$$r^{(s)} = \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 (z^{(s)} - z)^2} \quad (2)$$

which is the Euclidean distance between a receiver at position (x, y, z) and the satellite, s, at position $p^{(s)} = (x^{(s)}, y^{(s)}, z^{(s)})$. Note that this range $r^{(s)}$ is also time dependent, which is not explicitly stated in Equation 2. Obviously, the satellite moves and its position changes. But our receiver may also move (which is actually the interesting application for geophysics). Furthermore, both positions have to be given in the same coordinate system, which we will require to be earth centered earth-fixed, ECEF.

As an aside: this requires a rotation of the satellite position, from space-fixed into the earth-fixed coordinate system (otherwise approximately 10-20 m error) while also mapping the satellite position back to its location at send time and removing Earth's rotation rate (about 7×10^{-6} radians in the transmit time of 70-90 ms that it takes the signal to travel from satellite to receiver.):

$$p^{(s)} = \begin{bmatrix} cos(\omega_e \tau) & sin(\omega_e \tau) & 0 \\ -sin(\omega_e \tau) & cos(\omega_e \tau) & 0 \\ 0 & 0 & 1 \end{bmatrix} \widetilde{p}^{(s)} \tag{3}$$

where $\widetilde{p}^{(s)}$ is the satellite position in the space-fixed coordinate system, ω_e is Earth's rotation rate and τ remains the signal transit time. Again, we assume that we have the satellite position already given in ECEF.

4 Solving Pseudorange Measurement Model for Receiver Position

Replacing $r^{(s)}$ in Equation 1 with Equation 2 we get:

$$\rho^{(s)} = \sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2} + c\delta t_u - c\delta t^{(s)} + \epsilon \tag{4}$$

where $x, y, z, \delta t_u$ are unknown and ϵ now captures all delays including ionosphere and atmosphere delays given separately before. Unfortunately, Equation 4 is non-linear in x, y, z which prevents us from setting up a linear system of equations that could be easily solved with, for instance, least-squares approximations. What to do? Instead of throwing up our hands and walking away from the problem, we can try to find a linear approximation of the problem and solve that for receiver position and receiver clock error.

To achieve this, we will use the linear parts of a **multivariate Taylor Series expansion** of $\rho^{(s)}$, which assumes that we can approximate $\rho^{(s)}$ with a linear function in the vicinity of a point. For any function $f(x, y)$ that is at least differentiable once, a linear approximation about the point (a, b) is given by the sum of the function at this point and its partial derivatives at that point:

$$f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \tag{5}$$

If we linearize $\rho^{(s)}$ about an approximate position and expected receiver clock bias (x_0, y_0, z_0, t_{e_0}) using multivariate Taylor Series expansion, we get:

$$\rho^{(s)}(x, y, z, t_e) = \rho^{(s)}(x_0, y_0, z_0, t_{e_0}) + \frac{\partial \rho^{(s)}}{\partial x}(x - x_0) + \frac{\partial \rho^{(s)}}{\partial y}(y - y_0) + \frac{\partial \rho^{(s)}}{\partial z}(z - z_0) + \frac{\partial \rho^{(s)}}{\partial t_e}(t_e - t_{e_0}) + \epsilon \tag{6}$$

Keep in mind that we substituted δt_u with t_e to avoid double deltas here. We can simplify this a bit:

$$\begin{aligned} \rho^{(s)}(x, y, z, t_e) - \rho^{(s)}(x_0, y_0, z_0, t_{e_0}) &= \frac{\partial \rho^{(s)}}{\partial x}\Delta x + \frac{\partial \rho^{(s)}}{\partial y}\Delta y + \frac{\partial \rho^{(s)}}{\partial z}\Delta z + \frac{\partial \rho^{(s)}}{\partial t_e}\Delta t_e + \epsilon \qquad (7) \\ \Delta \rho^{(s)} &= \begin{bmatrix} \frac{\partial \rho^{(s)}}{\partial x} & \frac{\partial \rho^{(s)}}{\partial y} & \frac{\partial \rho^{(s)}}{\partial z} & \frac{\partial \rho^{(s)}}{\partial t_e} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix} + \epsilon \end{aligned}$$

Remember that $\Delta\rho^{(s)}$ is the difference between the measured pseudorange and the expected geometric range between a satellite position and the apriori position. We can calculate an updated absolute position and clock bias by adding $[\Delta x, \Delta y, \Delta z, \Delta t_e]$ to the apriori values $[x_0, y_0, z_0, t_{e_0}]$. We're pretty close to a solution here, but we first need to calculate the partial derivatives $\begin{bmatrix} \frac{\partial\rho^{(s)}}{\partial x} & \frac{\partial\rho^{(s)}}{\partial y} & \frac{\partial\rho^{(s)}}{\partial z} & \frac{\partial\rho^{(s)}}{\partial t_e} \end{bmatrix}$.

Let's work on this for the term $\frac{\partial\rho^{(s)}}{\partial x}$. We will need the chain rule:

$$\frac{\partial u^n}{\partial x} = n u^{n-1} \frac{\partial u}{\partial x} \tag{8}$$

and we set u to be the term under the square-root in the range expression in Equation 2:

$$u = (x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2 \tag{9}$$

we can write:

$$\begin{aligned}
\frac{\partial\rho^{(s)}}{\partial x} &= \frac{\partial\sqrt{(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2}}{\partial x} \qquad (10) \\
&= \frac{\partial\sqrt{u}}{\partial x} \\
&= \frac{1}{2}u^{-\frac{1}{2}}\frac{\partial u}{\partial x} \\
&= \frac{1}{2u^{\frac{1}{2}}}\frac{\partial[(x^{(s)} - x)^2 + (y^{(s)} - y)^2 + (z^{(s)} - z)^2]}{\partial x} \\
&= \frac{1}{2u^{\frac{1}{2}}}\frac{\partial[(x^{(s)} - x)^2]}{\partial x} \\
&= \frac{2(x^{(s)} - x)}{2u^{\frac{1}{2}}}(-1) \\
&= \frac{x - x^{(s)})}{\rho^{(s)}}
\end{aligned}$$

Doing this for all the partial derivatives at the apriori position gives us:

$$\frac{\partial\rho^{(s)}}{\partial x} = \frac{x_0 - x^{(s)}}{\rho_0^{(s)}} \tag{11}$$

$$\frac{\partial\rho^{(s)}}{\partial y} = \frac{y_0 - y^{(s)}}{\rho_0^{(s)}} \tag{12}$$

$$\frac{\partial\rho^{(s)}}{\partial z} = \frac{z_0 - z^{(s)}}{\rho_0^{(s)}} \tag{13}$$

$$\frac{\partial\rho^{(s)}}{\partial t_e} = c \tag{14}$$

Equation 15 follows from earlier expressions of δt_u. Note that $\rho_0^{(s)}$ is the geometric range from the apriori position to satellite s, which can be calculated without needing the precise position. Any model corrections that could be applied (e.g., troposphere, ...) could go in there, too. With these expressions for the partial derivatives, we can rewrite Equation 8:

$$\Delta\rho^{(s)} = \begin{bmatrix} \frac{x_0-x^{(s)}}{\rho_0^{(s)}} & \frac{y_0-y^{(s)}}{\rho_0^{(s)}} & \frac{z_0-z^{(s)}}{\rho_0^{(s)}} & c \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix} + \epsilon \tag{15}$$

Assuming that we have n satellites in view, each of which giving us pseudorange measurements $\rho^{(1)}, \ldots, \rho^{(n)}$, we can set up a linear system of equations:

$$\begin{bmatrix} \Delta\rho^{(1)} \\ \Delta\rho^{(2)} \\ \vdots \\ \Delta\rho^{(n)} \end{bmatrix} = \begin{bmatrix} \frac{x_0-x^{(1)}}{\rho_0^{(1)}} & \frac{y_0-y^{(1)}}{\rho_0^{(1)}} & \frac{z_0-z^{(1)}}{\rho_0^{(1)}} & c \\ \frac{x_0-x^{(2)}}{\rho_0^{(2)}} & \frac{y_0-y^{(2)}}{\rho_0^{(2)}} & \frac{z_0-z^{(2)}}{\rho_0^{(2)}} & c \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x_0-x^{(n)}}{\rho_0^{(n)}} & \frac{y_0-y^{(n)}}{\rho_0^{(n)}} & \frac{z_0-z^{(n)}}{\rho_0^{(n)}} & c \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix} + \epsilon \tag{16}$$

5 Solving the System

Equation 16 is of the form $Gm = d$ where G is the matrix with the partial derivatives, d is the vector with the pseudorange differences and m is the vector with the unknowns. We can solve this with least squares techniques to minimize th sum of squared residuals, for instance, using the normal equations:

$$m = (G^T G)^{-1} G^T d \tag{17}$$

We can also introduce a weight matrix W to, for instance, put less emphasis on satellites at low elevation angles:

$$m = (G^T W G)^{-1} G^T W d \tag{18}$$

Once we have a solution $m = [\Delta x, \Delta y, \Delta z, \Delta t_e]$ we can add these values to the apriori values to get an update:

$$\begin{bmatrix} x_{new} \\ y_{new} \\ z_{new} \\ t_{e_{new}} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ t_{e_0} \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_e \end{bmatrix} \tag{19}$$

and iterate until improvements are small.